International Journal of Applied Mathematics

Volume 25 No. 3 2012, 387-391

ON COMMUTATIVITY IN STRONGLY k-ENGEL π-REGULAR RINGS

A.Y.M. Chin[1], Sh. Sahebi[2 §]

[1]Institute of Mathematical Sciences
Faculty of Science
University of Malaya
50603, Kuala Lumpur, MALAYSIA
e-mail: acym@um.edu.my

[2]Department of Mathematics
Islamic Azad University
Tehran Centre, Tehran, IRAN
e-mail: sahebi@iauctb.ac.ir

Abstract: Let R be an associative ring with identity and let $k \geq 1$ be a fixed integer. An element $(x, y) \in R \times R$ is said to be left (right) k-Engel π-regular if there exist a positive integer n and an element $z \in R$ such that $[x, y]_k^n = z[x, y]_k^{n+1}$ ($[x, y]_k^n = [x, y]_k^{n+1}z$). If every element of $R \times R$ is left (right) k-Engel π-regular, then R is said to be left (right) k-Engel π-regular. An element $(x, y) \in R \times R$ is strongly k-Engel π-regular if it is both left and right k-Engel π-regular. The ring R is strongly k-Engel π-regular if every element of $R \times R$ is strongly k-Engel π-regular. In this paper we investigate conditions for a strongly k-Engel π-regular ring to be commutative.

AMS Subject Classification: 16U99, 16E50
Key Words: strongly π-regular ring, k-Engel ring

1. Introduction

Let R be an associative ring with identity. An element $x \in R$ is said to be right π-regular if there exist a positive integer n and an element $y \in R$ such

Received: April 7, 2012

§Correspondence author

that $x^n = x^{n+1}y$. If every element of R is right π-regular, then R is said to be right π-regular. By [3], this definition is left-right symmetric. An element of R is strongly π-regular if it is both left and right π-regular. R is strongly π-regular if every element of R is strongly π-regular. In [1] it was shown that if an element x in the ring R is strongly π-regular, then there exist a positive integer n and an element $y \in R$ such that $x^n = x^{n+1}y$ and $xy = yx$. In the case where $n = 1$, the element x is said to be strongly regular.

If $(x_i)_{i\in\mathbb{N}}$ is a sequence of elements of R and k is a positive integer, we define $[x_1, \ldots, x_{k+1}]$ inductively as follows:

$$[x_1, x_2] = x_1x_2 - x_2x_1,$$
$$[x_1, \ldots, x_k, x_{k+1}] = [[x_1, \ldots, x_k], x_{k+1}].$$

If $x_1 = x$ and $x_2 = \cdots = x_{k+1} = y$, the notation $[x, y]_k$ is used to denote $[x_1, \ldots, x_{k+1}]$ and $[x, y]_k$ is called a k-Engel element. For $k = 1$, $[x, y]_k = [x, y]_1$ is usually just denoted by $[x, y]$. We say that R satisfies the k-Engel condition if $[x, y]_k = 0$ for all $x, y \in R$. Commutative rings clearly have the k-Engel condition for any positive integer k.

An element $(x, y) \in R\times R$ is said to be left (right) k-Engel π-regular if there exist a positive integer n and an element $z \in R$ such that $[x, y]_k^n = z[x, y]_k^{n+1}$ ($[x, y]_k^n = [x, y]_k^{n+1}z$). If every element of $R\times R$ is left (right) k-Engel π-regular, then R is said to be left (right) k-Engel π-regular. An element $(x, y) \in R \times R$ is strongly k-Engel π-regular if it is both left and right k-Engel π-regular. The ring R is strongly k-Engel π-regular if every element of $R \times R$ is strongly k-Engel π-regular. Clearly, if (x, y) is strongly k-Engel π-regular, then $[x, y]_k$ is strongly π-regular. Therefore there exist a positive integer n and an element $z \in R$ such that $[x, y]_k^n = [x, y]_k^{n+1}z$ and $[x, y]_kz = z[x, y]_k$ (by [1]).

Division rings are examples of strongly k-Engel π-regular rings. Other examples include full matrix rings over division rings and triangular matrix rings over fields. It is clear that rings which satisfy the k-Engel condition are strongly k-Engel π-regular. In this paper we investigate conditions under which a strongly k-Engel π-regular ring is commutative (hence, has the k-Engel condition). All rings in this paper are assumed to have identity.

2. Main Results

Let k be a positive integer. We begin with the following characterization of strongly k-Engel π-regular rings.

Theorem 2.1. *Let R be a ring. Then R is strongly k-Engel π-regular if and only if for each $x, y \in R$, $[x, y]_k = a + w$ where $a, w \in R$ such that a is strongly regular, w is nilpotent and $aw = wa$.*

Proof. Suppose first that R is strongly k-Engel π-regular. Then there exist a positive integer n and an element $z \in R$ such that $[x, y]_k^n = [x, y]_k^{n+1} z$ and $[x, y]_k z = z[x, y]_k$. Since $[x, y]_k^n = [x, y]_k^{n+1} z = [x, y]_k^{n+2} z^2 = \cdots = [x, y]_k^{2n} z^n$, we have that

$$\begin{aligned}
([x, y]_k^{n+1} z^n)^2([x, y]_k^{n-1} z^n) &= [x, y]_k^{3n+1} z^{3n} \\
&= [x, y]_k^{n+1}([x, y]_k^{2n} z^n) z^{2n} \\
&= [x, y]_k^{2n+1} z^{2n} \\
&= [x, y]_k([x, y]_k^{2n} z^n) z^n \\
&= [x, y]_k^{n+1} z^n,
\end{aligned}$$

that is, $[x, y]_k^{n+1} z^n$ is strongly regular. Moreover, since $[x, y]_k^n z^n$ is an idempotent, we also have

$$\begin{aligned}
([x, y]_k - [x, y]_k^{n+1} z^n)^n &= [x, y]_k^n (1 - [x, y]_k^n z^n)^n \\
&= [x, y]_k^n (1 - [x, y]_k^n z^n) = 0.
\end{aligned}$$

Thus, $[x, y]_k - [x, y]_k^{n+1} z^n$ is nilpotent. It is clear that

$$[x, y]_k^{n+1} z^n([x, y]_k - [x, y]_k^{n+1} z^n) = ([x, y]_k - [x, y]_k^{n+1} z^n)[x, y]_k^{n+1} z^n.$$

Then since $[x, y]_k = [x, y]_k^{n+1} z^n + ([x, y]_k - [x, y]_k^{n+1} z^n)$, the result follows.

Conversely, let $x, y \in R$ and suppose that $[x, y]_k = a + w$ where a is strongly regular, w is nilpotent and $aw = wa$. Since a is strongly regular, there exists $b \in R$ such that $a^2 b = a$, $ab = ba$ and $bw = wb$ (by [1, Lemma 1]). Since $[x, y]_k - [x, y]_k^2 b = (a + w) - (a + w)^2 b = w - w(2a + w)b$ is nilpotent and $[x, y]_k b = b[x, y]_k$, it follows that $[x, y]_k^n = [x, y]_k^{n+1} z$ for some positive integer n and some $z \in R$ with $[x, y]_k z = z[x, y]_k$. Thus R is strongly k-Engel π-regular. □

As a consequence of Theorem 2.1, we have the following:

Corollary 2.2. *Let R be a reduced ring. If R is strongly k-Engel π-regular, then one of the following occurs:*

(a) *R has the k-Engel condition;*

(b) *There exist $x, y \in R$ such that $[x, y]_k = a$ for some non-zero strongly regular element a.*

Proof. Since R is reduced, the only nilpotent element in R is the zero element. Thus by Theorem 2.1, for any $x, y \in R$, $[x, y]_k = a$ for some strongly regular element $a \in R$. If R has no non-zero strongly regular element, then R satisfies the k-Engel condition. Otherwise, we have condition (b) as asserted. This completes the proof. $\square$

Corollary 2.3. *Let R be a reduced strongly k-Engel π-regular ring. Then either R is commutative or there exist $x, y \in R$ such that $[x, y]_k = a$ for some non-zero strongly regular element a.*

Proof. This follows by Corollary 2.2 and the fact that a reduced ring with the k-Engel condition is commutative (see [2, Lemma 18]). $\square$

For a ring R, let $D(R)$ denote the commutator ideal of R, that is, the ideal generated by all commutators of R. We next consider a stronger condition than that in Corollary 2.2, namely, when the ring has no non-zero divisors.

Proposition 2.4. Let R be a strongly k-Engel π-regular ring. If R has no zero divisors, then $R = D(R)$ or R is commutative.

Proof. Let $x, y \in R$. Since R is strongly k-Engel π-regular, there exist $z \in R$ and a positive integer n such that $[x, y]_k^n = [x, y]_k^{n+1} z$ and $[x, y]_k z = z[x, y]_k$. Then $[x, y]_k^n(1 - [x, y]_k z) = 0$. Since R has no zero divisors, it follows that $[x, y]_k = 0$ or $[x, y]_k z = 1$. If $R \neq D(R)$, then $[x, y]_k z \neq 1$ for any $x, y, z \in R$. Hence, $[x, y]_k = 0$ for all $x, y \in R$ and it thus follows by [4, Theorem 1.2] that R is commutative. $\square$

Remark. The semiprimitive strongly k-Engel π-regular rings are not necessarily commutative. For example, the ring $M_2(\mathbb{F}_2)$ of 2×2 matrices over the field $\mathbb{F}_2$ is a semiprimitive strongly k-Engel π-regular ring which is not commutative.

For a ring R, let $N_r(R)$, $Z(R)$ and $J(R)$ denote the sum of all the nil right ideals of R, the center of R and the Jacobson radical of R, respectively. It is known that $N_r(R)$ is also the sum of all the nil left ideals of R and that $N_r(R)$ is thus a two-sided ideal of R.

Proposition 2.5. *Let R be a strongly k-Engel π-regular ring with $N_r(R) = 0$. If for any $x, y \in R$ there exists a positive integer $t = t(x, y)$ such that $(xy)^t - (yx)^t \in Z(R)$, then R is commutative.*

Proof. Let $x, y \in R$. Since R is strongly k-Engel π-regular, there exist $z \in R$ and a positive integer n such that $[x, y]_k^n = [x, y]_k^{n+1} z = [x, y]_k^n [x, y]_k z$ and $[x, y]_k z = z[x, y]_k$. But, since $(xy)^t - (yx)^t \in Z(R)$ for some positive integer t, we have that $[x, y]_k z \in J(R)$ (by [5, Lemma 3]). Hence, $1 - [x, y]_k z$ has a right inverse and since $[x, y]_k^n (1 - [x, y]_k z) = 0$, we have that $[x, y]_k^n = 0$. It then follows by [4, Theorem 1.2] that R is commutative. $\square$

References

[1] G. Azumaya, Strongly π-regular rings, *J. Fac. Sci. Hokkaido Univ. Ser. I*, **13** (1954), 34-39.

[2] C.L. Chuang, J.S. Lin, On a conjecture by Herstein, *J. Algebra*, **126** (1989), 119-138.

[3] M.F. Dischinger, Sur les anneaux fortement π-réguliers, *C.R. Acad. Sci. Paris Sér. A–B*, **283** (1976), 571-573.

[4] A. Giambruno, J.Z. Gonçalves, A. Mandel, Rings with algebraic n-Engel elements, *Comm. Algebra*, **22** (1994), 1685-1701.

[5] A.H. Yamini, Sh. Sahebi, Rings satisfying the generalized polynomial identity $(x - x^n)([x, y]_k - [x, y]_k^m) = 0$, *Riv. Mat. Univ. Parma*, **6** (1999), 11-18.